Introduction to Mathematical Neuroscience

Kenneth D. Swegman Jr.
Department of Mathematics
La Roche College

December 28, 2021

1 Appendix

Contents

2 Acknowledgments

Thanks to my Mom, Dad, and my little Sister for the help and mental strength to write this.

Also thanks to Dr. O'Grady for being there much of my Mathematical career, and for this class as well.

Thanks to all my friends in the Comp Sci. program at La Roche college.

Thanks to my friends, and my internship at Duquense Light Company, that taught me so much about the field of Computer Science.

I'd like to personally shoutout my friends who have helped me out here and there and have taught me much along the way.

My friends Connor, CJ, Joey, who have been very impactful for me wanting to learn.

They have been great sources of knowledge for me.

Thank you! To everyone I did not shoutout that helped me in someway.

3 Lecture 1, May 21st

Goals for the Class:

- Learn Fundamental Mathematical Neuroscience
- Learn Code to Solve and Analyze differential equations
- Learn LaTex

Before we start to model the Brain, Let's review some math.

Many of the equations we will study, must be analyzed/solved using computers.

To Motivate, Recall 2 famous problems from mathematics:

1. Equation Solving
2. Integration

Many equations can be solved by hand symbolically

For example,

$$3x + 5 = 6$$

and

$$5x^2 - 3x + 10 = 0$$

The key observation is that we can solve them exactly!

However, we can NOT Solve

$$5x^7 + 3x^3 + 6 = 0$$

By Symbolic means

We can Approximate a Solution or solutions using a calculator, graph or Newton's Method

Integration is a process defined as a complicated limit:

$$\int_a^b f(x)dx = \lim_{n \to \infty} f(x) \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Fundamental theorem of Calculus

The FTC gives a Symbolic Method for computing <u>Some</u> Integrals

For Example,

$$\int_1^3 x^2 dx = \frac{1}{3}x^3\Big|_1^3$$
$$= \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3$$
$$= \frac{27}{3} - \frac{1}{3}$$
$$= \frac{26}{3}$$

<u>BAD NEWS:</u> The FTC only works if f(x) is continuous and we know the anti-derivative

We knew

$$\frac{d}{dx}(\frac{1}{3}x^3) = x^2$$

This means integrals such as

$$\int_0^\pi \frac{sinx}{x}dx$$

We can use other methods to approximate the value:

1. Numerical Integration (Mid-Point Rule, Trapezoidal Rule, Simpson's Rule)
2. Power Series
$$\frac{sinx}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$$

The same issue arrives when studying Differential Equations

Some can be solved symbolically; many can't.

To illustrate How we might proceed.

Consider the equation

$$\frac{dy}{dx} = 5y$$

with

Initial Condition $y(0) = 2$

If you don't see that $y = Ae^{5x}$ is the General Solution

We can Derive it by Separating the Variables:

$$\frac{dy}{dx} = 5y$$

$$\Longrightarrow \frac{dy}{y} = 5dx$$

$$\Longrightarrow \int \frac{1}{y} dy = \int 5dx$$

$$\Longrightarrow ln|y| = 5x + c \Rightarrow y = e^{5x+c}$$

or

$$y = Ae^{5x}$$

Let's determine A by using $y(0) = 2$:

$$y(0) = Ae^0 = A; \text{ and } y(0) = 2$$

Thus, by the transitive property,

$$A = 2$$

Particular Solution is

$$y(x) = 2e^{5x}$$

An alternative method is to solve

$$\frac{dy}{dx} = 5y$$

"Numerically" using Euler's Method (Oiler).

The idea is to approximate the Solution curve using a Discretized Domain

Recall

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

If h $\approx$ 0, then

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h}$$

Let's plug this into the Eq.

$$\frac{dy}{dx} = 5y$$

$$\frac{y(x+h) - y(x)}{h} x 5y(x)$$

Next solve for $y(x+h)$:

$$y(x+h) \approx 5y(x) * h + y(x)$$

Since $y(0) = 2$ we can compute $y(h)$ using

$$y(h) \approx 5y(0) * h + y(0)$$
$$= 10h + 2$$

Let's use the "domain"

$$[0, .05, .1, .15, .2, ..., 2]$$

We can determine the corresponding outputs using **.

Set $h = .05$ and $x_0 = 0$

$$x_1 = 0 + h$$
$$x_2 = x_1 + h$$
$$.$$
$$x_i = ih$$
$$.$$
$$.$$
$$x_i + 1 = x_i + h$$

We can compute

$$y(x_1) = 5y(x_0)h + y(x_0)$$

$$y(x_2) = 5y(x_1)h + y(x_1)$$

$$.$$

$$.$$

$$y(x_i + 1) = 5y(x_i)h + y(x_i)$$

$$= 5y(x_i)(.05) + y(x_i)$$

Let's make a Chart

x_i	$y(x_i)$
$x_0 = 0$	$y(0) = 2$
$x_1 = .05$	$y(.05) = 2.5$
$x_2 = .1$	$y(.1) = 3.125$
$x_3 = .15$	$y(.15) = 3.90625$

End of Lecture I

4 Lecture 2, May 23rd

<u>Grade</u>
Practice Sets - 10 Points each
4 Projects - 100 Points each
Participation - 25 Points (Class, and Notes)

Today we solve a few more diff eqs. by hand and get back to Euler's Method.

Practice Set 1

2B.

$$\frac{dy}{dt} = y(1+t^2), y(0) = 3$$

Seperable Eq.

This eq. is seperable, because we can rewrite as

$$\frac{dy}{y} = (1+t^2)dt$$

Next, we Integrate both sides:

$$\int \frac{1}{y}dy = \int 1 + t^2 dt$$

$$\longrightarrow ln|y| = t + \frac{1}{3}t^3 + c$$

Next, solve for y:

$$y(t) = e^{t+\frac{1}{3}t^3+c}$$

or

$$y(t) = Ae^{t+\frac{1}{3}t^3}$$

Note: $e^c = A$

To finish we use the initial condition $y(0) = 3$ to Determine A.

$$y(0) = Ae^{0+0} = A$$

$$\text{and}$$

$$y(0) = 3$$

So that $A = 3$

The Particular Solution is

$$y(t) = 3e^{t+\frac{1}{3}t^3}$$

Note: This is an Exact Solution to the Initial Value Problem (IVP).

Practice Set 1 Problem 2(d)

The equation

$$t\frac{dy}{dt} + 2y = t^2 - t + 1$$

is <u>NOT</u> Seperable.

However, if we divide through by t we obtain

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

All t's set form

General form of Eq.

Which is of the form

$$y' + p(t)y = q(t)$$

First form linear eq.

$$Derivative + Function = Somethingelse$$

Equations of this form are called

<u>1st Order linear equations</u>

To solve eqs. of this type, we use the so-called

Integrating Factor.

The Integrating Factor is

$$I(t) = e^{\int p(t)dt}$$
$$I(t) = e^{\int \frac{2}{t}dt}$$
$$I(t) = e^{2*lnt}$$
$$I(t) = e^{lnt^2}$$
$$I(t) = t^2$$

(Ignore Constant c)

D maps function to function

$$D(y) = t * y' + 2y$$
$$D(y) = t^2 - t + 1$$

$$D(cy_1 + y_2) = ... = cD(y_1) + D(y_2)$$

Next, Multiply both sides by $I(t) = t^2$:

$$t^2(\frac{dy}{dt} + \frac{2}{t}y) = t^2(t - 1 + \frac{1}{t})$$

$$\to t^2\frac{dy}{dt} + 2t * y = t^3 - t^2 + t$$

Key observation is that the left hand side is a derivative!!

$$(t^2 + y)' = t^3 - t^2 + t$$

To get y we Integrate and obtain

$$t^2 y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$

$$\to y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + ct^{-2}$$

End of Lecture II

5 Lecture 3, May 30th

The 1st two lectures we reviewed some basic 1st order differential equations.

The reason is because one of the most elementary Neuronal models is this type of eq.:

$$V'(t) + \frac{1}{\tau}V(t) = f(t)$$

$$V(0) = b$$

Here, τ (Greek Tau) is a constant.

V(t) represents Voltage Difference

We will Derive this in Detail soon enough.

Stay tuned for what this means in Detail.

Just as we did last week, we can solve this using an Integrating Factor.

<u>Case1 ($\tau = \infty$)</u>

eq. reduces to

$$V'(t) = f(t)$$

and thus, By FTC

$$\int_0^T V'(t)dt = \int_0^T f(t)dt \Longrightarrow V(t) - V(0) = \int_0^T f(t)dt$$

$$\Longrightarrow V(t) = b + \int_0^T f(t)dt$$

<u>Case2 ($\tau > 0$)</u>

$$V'(t) + \frac{1}{\tau}V(t) = f(t)$$

Here, Integrating factor is

$$e^{\int \frac{1}{t}dt} = e^{\frac{1}{\tau}t}$$

Multiply both sides by

$e^{\frac{t}{\tau}}$ to obtain

$$[V'(t) + \frac{1}{\tau}V(t)]e^{\frac{t}{\tau}} = f(t)e^{\frac{t}{\tau}}$$

Recognize the LHS as a Derivative

$$[V(t)e^{\frac{t}{\tau}}]' = f(t)e^{\frac{t}{\tau}}$$

Integrate

$$\int_0^T [V(t)e^{\frac{t}{\tau}}]'dt = \int_0^T f(t)e^{\frac{t}{\tau}}$$

$$\implies V(T)e^{\frac{T}{\tau}} - V(0)e^0 = \int_0^T f(t)e^{\frac{t}{\tau}}dt$$

$$\implies V(T)e^{\frac{T}{\tau}} - b = \int_0^T f(t)e^{\frac{t}{\tau}}dt$$

Lastly, Solve for V(T)

$$V(T) = be^{\frac{T}{\tau}} + e^{-\frac{T}{\tau}}\int_0^T f(t)e^{\frac{t}{\tau}}dt$$

Or

$$V(t) = be^{-\frac{T}{\tau}} + \int_0^T f(t)e^{\frac{1}{\tau}(t-T)}dt$$

It is useful to have this formula handy!

Okay, What do all these unknowns represent?

Well, $V(0) = b = V_{CL}$

$$f(t) = \frac{V_{CL}}{\tau} + \frac{I_{stim}(t)}{AC_m}$$

Where V_{CL}, τ, AC_m are constants determined by experiment

To get an idea what solutions may look like assume we have a simple situation where

$V_{CL} = -70, \tau = .5, AC_m = 1$, and

$$\begin{aligned} I_{stim}(t) = 0 \qquad & 0 < t < 3 \\ = 1 \qquad & 3 \le t \le 4 \\ = 0 \qquad & t > 4 \end{aligned}$$

Well,

$$V(T) = V_{CL}e^{-\frac{T}{\tau}} + \int_0^T f(t)e^{\frac{(t-T)}{\tau}}dt$$

$$= -70e^{-2T} + \int_0^T [-2 * 70 + I_{stim}(t)]e^{2(t-T)}dt$$

$$= -70e^{-2T} + \int_0^T -140e^{2(t-T)}dt + \int_0^T I_{stim}(t)e^{2(t-T)}dt$$

$$= -70e^{-2T} - 140e^{-2T}\int_0^T e^{2t}dt + \int_0^T I_{stim}(t)e^{2(t-T)}dt$$

$$= -70e^{-2T} - 140e^{-2T}(\frac{e^{2T}}{2}\Big|_0^T) + \int_0^T I_{stim}(t)e^{2(t-T)}dt$$

$$= -70e^{-2T} - 140e^{-2T}(\frac{e^{2T}}{2} - \frac{1}{2}) + \int_0^T I_{stim}(t)e^{2(t-T)}dt$$

$$= -70e^{-2T} - 70e^{-2T}(e^{2T-1} - 1) + \int_0^T I_{stim}(t)e^{2(t-T)}dt$$

$$\implies V(t) = -70 + \int_0^T I_{stim}(t)e^{2(t-T)}dt$$

3 Cases to Consider:

<u>Case1 ($T \leq 3$)</u>

Here, $I(t) = 0$ Why?

By definition of $I(t)$!

$V(T) = -70$ to since

$$\int_0^T I(t)e^{2(t-T)}d = \int_0^T 0 * e^{2(t-T)}dt = 0$$

Why??

Since $T \leq 3, \quad 0 \leq t \leq 3$

<u>Case2 ($3 \leq T \leq 4$)</u>

Thus,

$$\int_0^T I(t)e^{2(t-T)}dt = ?$$

Well,

Sketch

$$\begin{aligned} I(t)e^{2(t-T)} &= 0 \quad 0 \leq t \leq 3 \\ &= e^{2(t-T)} \quad 3 \leq t \leq 4 \\ &= 0 \qquad t > 4 \end{aligned}$$

Thus,

$$\int_0^T I(t)e^{2(t-T)}dt = \int_0^3 I(t)e^{2(t-T)}dt + \int_3^T I(t)e^{2(t-T)}db$$

$$= \int_3^T e^{2(t-T)} dt = e^{-2T} \int_3^T e^{2T} dt$$

$$= e^{-2T} (\frac{1}{2} e^{2t})^T$$

$$= \frac{e^{-2T}}{2} (e^{2T} - e^6)$$

$$= \frac{1}{2} (1 - e^{6-2T})$$

Thus,

$$V(T) = -70 + \tfrac{1}{2}(1 - e^{6-2T}) \text{ if } 3 \leq T \leq 4$$

<u>Case3 ($T > 4$)</u>

Here,

$$V(T) = -70 + \int_0^T I(t) e^{2(t-T)} dt$$

$$= -70 + \int_0^3 I(t) e^{2(t-T)} dt + \int_3^4 I(t) e^{2(t-T)} dt + \int_4^T I(t) e^{2(t-T)} dt$$

$$= -70 + \int_3^4 e^{2(t-T)} dt$$

$$= -70 + e^{-2T} (\frac{1}{2} e^{2T}) \Big|_3^4 = -70 + \frac{e^{-2T}}{2} (e^8 - e^6)$$

In Conclusion, the solution is

$$V(T) = -70 \quad 0 \leq T \leq 3$$
$$= -70 + \frac{1}{2}(1 - e^{6-2T}) \quad 3 \leq T \leq 4$$

$$= -70 + \frac{e^{-2t}}{2}(e^8 - e^6) \quad T > 4$$

End of Lecture III

6 Lecture 4, June 4th

Last class we studied the IVP(Initial Value Problem)

$$V'(t) + \frac{1}{t}V(t) = f(t)$$
$$V(0) = b$$

Where $b = V_{CL}$ and

$$f(t) = \frac{1}{t}V_{CL} + \frac{I_{stim}(t)}{AC_m}$$

The problem is easy if we sent $\tau \to \infty$

If not, the problem is dependent on

$$I_{stim}(t)$$

We then solve the IVP in a specific situation. If we repeated that argument with constant we can determine the precise solution.

For a Square Impulse we can write down the explicit solution.

The characteristic function of a set A is

$$\mathbb{1}_A(t) = 0 \text{ if } t \not\in A$$

$$= 1 \text{ if } t \in A$$

Suppose (t_1, t_2) is any Interval. Then,

$$\mathbb{1}_{(b,t_2)}(t) = 0 \text{ if } t \not\in (t_1, t_2)$$

$$= 1 \text{ if } t \in (t_1, t_2)$$

Here's the idea: A total charge of Q is Given to the cell from $t_1 = t_1$ seconds

Until $t = t_2$ seconds.

Then,

$$I_{stim}(t) = \frac{Q}{t_2 - t_1} \mathbb{1}_{(t_1,t_2)}(t)$$

If $Q = 3$, $t_1 = 2$, and $t_2 = 8$

We have

If $2 < t < 8$, $I_{stim}(t) = \frac{Q}{6} * 1$ if $Q = 3$, $I_{stim}(t) = \frac{1}{2}$

This is called a Square Impulse.

The Precise Solution for V is

$$V(t) = V_{CL} + \frac{Q\tau}{(t_1, t_2)AC_m}$$

$$= 0 \text{ if } t < t_1$$

$$= 1 - e^{\frac{(t_1 - t_2)}{\tau}} \text{ if } t_1 \leq t \leq t_2$$

$$= e^{\frac{1}{\tau}(t_2 - t)} - e^{\frac{1}{\tau}(t_1 - t)} \text{ if } t > t_2$$

<u>Sinusoidal Input</u>

Here, $I_{stim}(t) = I_0 \sin(2\pi wt)$

Where I_0 and w are constants.

Well,

$$V(T) = V_{CL} + \frac{1}{AC_m} \int_0^T e^{\frac{(t-T)}{\tau}} I_{stim}^{(t)} dt$$

$$= V_{CL} + \frac{I_0}{AC_m} \int_0^T e^{\frac{1}{\tau}(t-T)} \sin(2\pi wt) dt$$

$$= V_{CL} + \frac{I_0}{AC_m} e^{-\frac{T}{\tau}} \int_0^T e^{\frac{1}{\tau}t} \sin(2\pi wt) dt$$

$$= V_{CL} + \frac{I_0}{Ac_m} e^{-\frac{T}{\tau}} \Big[\frac{e^{\frac{1}{\tau}}t}{(\frac{1}{\tau})^2 + (2\pi w)^2} (\frac{1}{\tau} \sin(2\pi wt) - 2\pi w \cos(2\pi w \cos(2\pi wt)) \Big]_0^T$$

$$= V_{CL} + \frac{I_0 e^{-\frac{T}{\tau}}}{AC_m(\frac{1}{\tau^2} + 4\pi^2 w^2)} * [e^{\frac{T}{\tau}}(\frac{1}{\tau}\sin(2\pi wt) - 2\pi w\cos(2\pi wT)) - e^0(\frac{1}{\tau} * \sin(0) - 2\pi w\cos(0))]$$

$$= V_{CL} + \frac{I_0 e^{-\frac{T}{\tau}}}{AC_m(\frac{1}{\tau^2} + 4\pi^2 w^2)} * [e^{\frac{T}{\tau}}(\frac{1}{\tau}\sin(2\pi wT) - 2\pi w\cos(2\pi wT)) - (0 - 2\pi w)]$$

$$= V_{CL} + \frac{I_0}{AC_m(\frac{1}{\tau^2} + 4\pi^2 w^2)} * (BLAH),$$

$$BLAH = 2\pi w e^{-\frac{T}{\tau}} + \frac{\sin(2\pi wT)}{\tau} - 2\pi w\cos(2\pi wT)$$

Yikes!!!

Let's graph for $V_{CL} = -70$

$I_0 = 1$, $AC_m = 2$, $\tau = \frac{1}{2}$, $w = \frac{1}{2\pi}$:

$$V(T) = -70 + \frac{1}{2 * (4+1)}[e^{-2T} + 2\sin(T) - \cos(T)]$$

What if $I_0 = 5$ and $\tau = 1$:

$$V(T) = -70 + \frac{5}{2(1+1)}[e^{-T} + \sin(T) - \cos(T)]$$

If $\tau = 5 : -2\pi w\cos(2\pi wT)$

$$-70 + \frac{5}{2(\frac{1}{25}+1)}\left[e^{-\frac{T}{5}} + \frac{\sin(T)}{5} - \cos(T)\right]$$

End of Lecture IV

7 Lecture 5, June 6th

Today we finally show how to write code MATLAB/Octave code.

Before discussing this Let's discuss proper notation. It is convenient to rewrite our ODEs (Ordinary Differential Equations) In the form

$$\frac{dy}{dt} = F(y,t)$$

Let's do this for the problems in Practice Set 1

(a.) $\frac{dy}{dt} = F(y,t)$ where $F(y,t) = yt$

(b.) $\frac{dy}{dt} = F(y,t)$ where $F(y,t) = y(1+t^2)$

(c.) $\frac{dy}{dt} = G(y)$ where $G(y) = 9.8 - .15y$

Note: RHS does not depend on time.

(d.) $\frac{dy}{dt} = G(y,t)$ where $G(y,t) = t - 1 + \frac{1}{t} - \frac{2}{t}y$

(e.) $\frac{dv}{dt} = H(v,t)$ where $H(v,t) = -1400 + te^{-t} - \frac{1}{.05}v$

Using this form of the ODE we can easily use Discrete times to compute an approximated solution

To be more specific:

Instead of solving

$$\frac{dy}{dt} = F(y,t)$$

For all t in same interval

We will solve for

$$t_1, t_2, t_3, t_4, ..., t_n \text{ and obtain}$$

$$y(t_1), y(t_2), ..., y(t_n)$$

We use the definition of the derivative and note

$$\frac{dy}{dt} \approx \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}$$

So that if $\Delta t = t_i + 1 - t_i$ we have

$$\frac{y(t_{i+1}) - y(t_i)}{\Delta t} \approx F(y(t_i), t_i)$$

We can rewrite as

$$y(t_{i+1}) - y(t_i) + \Delta t F(y, t)$$

So, if we know $y(t_1)$ then we can readily compute $y(t_2), y(t_3), ...$

This is called the Forward Euler Method.

This is too tedious to do by hand so we have the computer do the work.

Some MatLab Basics

1. Matlab likes to store "things" as vectors. For example, we can let

$$t = (t_1, t_2, t_3, ..., t_n)$$

and

$$y = (y(t_1), y(t_2), ..., y(t_n))$$

2. If we have 2 vectors t and y we can graph them with

$$\text{Plot}(t, y)$$

$$F = inline('t - 1 + (1./t) - 2./t).'y',' y',' t');$$

$$N = 500$$

$$dt = .001$$

$$t = Zeros(1, N);$$

$$y = Zeros(1, N);$$

$$t(1) = 1;$$
$$y(1) = .5;$$

$$\text{for } i = 1; N - 1;$$

$$y(i+1) = y(i) + dt * f(y(i), t(i));$$
$$t(i+1) = t(i) + dt;$$

$$\text{end}$$

$$t_1 = 0.1 : 0.1 : t(N)$$

yexact

Seperable Equation

$$\frac{dy}{dt} = yt <==> \frac{dy}{y} = tdt$$

$$<==> lny = \frac{1}{2}t^2 + d$$

$$==> y = Ae^{\frac{1}{2}t^2}$$

$$y(0) = A = 1 ==> A = 1$$

End of Lecture V

8 Lecture 6-10, June 11th-June 20th

These lectures we worked on MatLab/Octave Code, and also worked on Overleaf LaTex during class time.

9 Lecture 11, June 25th

Summarize Course

- Review of ODEs (Seperable, and 1st Order linear)
- Stated and Solved Basic Neuron Model
- Introduced LaTex

Today we will discuss elements from linear algebra used in Programming

Then we Discuss Programming in MatLab lecture.

Recall that A matrix $(m * n)$ B_{m*n} is an array of numbers written as

$$B = \begin{pmatrix} b_{11} & b_{12} & .... & b_{1n} \\ b_{21} & b_{22} & .... & b_{2n} \\ . & . & & \\ . & & . & \\ . & & & . \\ b_{m1} & b_{m2} & .... & b_{mn} \end{pmatrix}$$

<u>Basic Examples</u>

$$B = \begin{pmatrix} 1 & 5 \\ \frac{3}{2} & 6 \end{pmatrix} \quad 2x2$$

$$A = \begin{pmatrix} 5 & 6 & 10.3 & 5 \\ 4 & -3 & .5 & 6 \end{pmatrix} \quad 2x4$$

$1 * n$ matrix is often called a <u>Row Vector</u>.

A $m * 1$ matrix is often called a <u>Column Vector</u>.

<u>Examples</u>

$\begin{pmatrix} 1 & 6 & 7 & 8 \end{pmatrix}$ is a row vector.

$\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$ is a column vector.

Given 2 Matrices of the same dimension we can define addition and subtraction.

$$A_{m*n} = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ . & . & & \\ . & & . & \\ . & & & . \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}$$

$$B_{m*n} \begin{pmatrix} b_{11} & b_{12} & .... & b_{1n} \\ . & . & & \\ . & & . & \\ . & & & . \\ b_{m1} & b_{m2} & .... & b_{mn} \end{pmatrix}$$

$$C = A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & .... & a_{1n} + b1n \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} + b_{m1} & .... & .... & a_{mn} + b_{mn} \end{pmatrix}$$

<u>Example</u>

$$\begin{pmatrix} 4 & 5 & 6 \\ -2 & 3 & 25 \end{pmatrix} + \begin{pmatrix} 10 & -5 & 12 \\ -1 & 15 & -6 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 18 \\ -3 & 18 & 19 \end{pmatrix}$$

Subtraction is Defined in obvious way.

Multiplication is much more complicated

Start with 2 Vectors

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & .... & x_n \end{pmatrix}$$
$$y = \begin{pmatrix} y_1 & y_2 & y_3 & .... & y_n \end{pmatrix}$$

We define the <u>Scalar</u> or <u>Dot Product</u> to be

$$x * y = x_1y_1 + x_2y_2 + ... + x_ny_n = \sum_{i=1}^{\hat{}} x_iy_i$$

Example

$$x = \begin{pmatrix} 1 & -3 & 10 \end{pmatrix}$$
$$y = \begin{pmatrix} 6 & 4 & -2 \end{pmatrix}$$

$$\begin{aligned} x * y &= 1 * 6 + (-3) * 4 + 10(-2) \\ &= 6 - 12 - 20 \\ &= -26 \end{aligned}$$

Note: The dot product of two vectors is a real number

There are other useful ways to define multiplication of vectors, but they're not useful to this class.

The norm of a vector is denoted by

$$||x|| = \sqrt{x * x} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

This is just a generalization of the distance formula AKA the Pythagorean Theorem.

To see this think about the "vector" $(x_1 x_2 x_3)$ as a point in 3D Space.

Think about $x = (x_1 x_2)$ as a "Point" in the plane!

Rest of the day practiced MatLab

Addition Notes from MatLab Lecture

Practice Set 2

1. Learn how to Plot on Octave/MatLab
2. Then do numeric Integration.
3. Then Euler Method

MidPoint

$$[x_1, x_1], [x_1, x_2], ...[x_{i-1}, x_i]$$

$$m_i = \frac{x_{i-1} + x_i}{2}$$

$$m = \begin{pmatrix} m_1 & m_2 & .... & m_n \end{pmatrix} \quad = \begin{pmatrix} m(1) & m(2) & .... & m(n) \end{pmatrix}$$

$$m(1) = \frac{1}{8}$$
$$m(2) = \frac{3}{8}$$
$$m(3) = \frac{5}{8}$$
$$m(y) = \frac{7}{8}$$

$$m(1) = \frac{0 + \frac{1}{4}}{2}$$

$$m(2) = \frac{\frac{1}{4} + \frac{2}{4}}{2}$$

Recall: Left Sum

$$\sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$= \Delta x[f(x_0) + f(x_1) + ... + f(x_{n-1})]$$

Right Sum

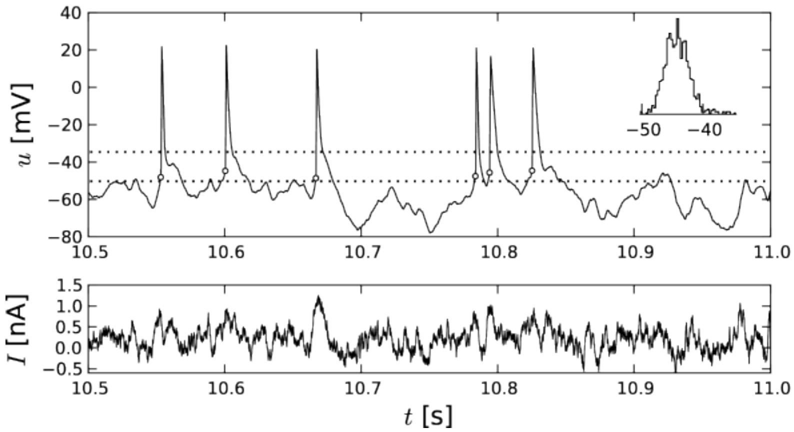
$$\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \Delta x[f(x_1) + ... + f(x_n)]$$

End of Lecture XI

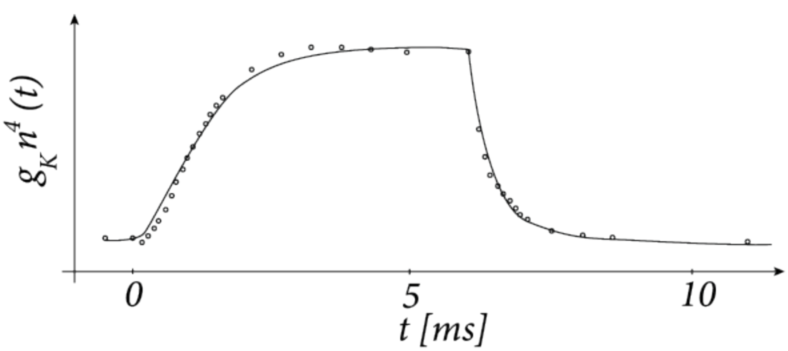
10 Lecture 12, June 27th

Recall that artain "neuron" experiments suggest that the "voltage" of a given neuron exhibits the following behavior



Spike Train Version, but still stochastic and more realistic.

For "Low" constant current the experiment shows



Our job is to develop and study mathematical models (Differential eqs.) that "describe" the experiements.

Early this term we stated a model called the passive cell model:

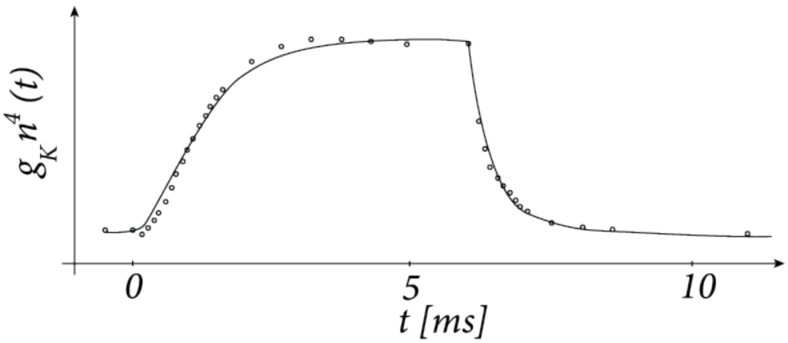
$$\frac{dv}{dt} = -\frac{1}{\tau}V + \frac{V_{CL}}{\tau} + \frac{I(t)}{A * C_m}$$

Where τ, V_{CL}, A, and C_m are constants.

We solve this by hand for specific currents I(t).

If we apply a Constant Current over a time interval $[t_1 \ t_2]$

We obtain



t_1 is at the start point of 0 right before the current spike. t_2 is at around 7 where it starts to die off, and meets back to where it rests.

Using Octave/MatLab we can experiment with all types of current input.

Before discussing a more realistic model, let's dig deeper into the Passive Model.

$$I(i) = (t(i) > 2) * (t(i) < 22) * 1e - 5$$

Our next goal is to study system of ODEs.

Recall the idea for one equation.

$$\frac{dy}{dt} = F(v, t)$$

We use

$$\frac{dv}{dt} \approx \frac{V(t_{i+1}) - V(t_i)}{\Delta V}$$

To Write

$$V(t_{i+1} \approx V(t_i) + \Delta t F(V(t_i), t_i)$$

A more precuse formulation says to use MVT Theorem:

$$V(t_{i+1}) - V(t_i) = V'(C_i)\Delta t$$

Where

$$C_i \quad \epsilon \quad (t_i, t - i + 1)$$

Thus, if $\Delta\tau \approx 0, C_i \approx t_i$, and we have

$$V(t_{i+1} = V(t_i) + V'(t_i)\Delta t$$
$$V(t_{i+1} = V(t_i) + F(V(t_i), t_i)\Delta t$$

What about a system of equations?
Say

$$\frac{dV_1}{dt} = F_1(V_1, t)$$

$$\frac{dV_2}{dt} = F_2(V_2, t)$$

Applying Eulers Method to both we have

$$V_1(t_{i+1} = V_1(t_i) + \Delta t F_1(V_1(t_i), t_i)$$

$$V_2(t_{i+1} = V_2(t_i) + \Delta t F_2(V_2(t_i), t_i)$$

Key observation:

What if both equations depend on each other?

End of Lecture XII

11 Lecture 13, July 2nd

Recall that our Current goal is to study the famous

<u>Hodgkin-Huxley Neuron Model</u>

This model is a <u>System of Differential Equations</u>.

A first order system is any system of the form

$$\frac{dx_1}{dt} = F_1(t, x_1, x_2, ..., x_n)$$

$$\frac{dx_2}{dt} = F_2(t, x_1, x_2, ..., x_n)$$

Example:

$$\frac{dx_1}{dt} = 3x_1 + 5t$$

$$\frac{dx_2}{dt} = 5x_1x_2$$

Here,

$$F_1 = 3x_1 + 5t$$
$$F_2 = 5x_1x_2$$

In Vector Notation we just write

$$\frac{d\vec{x}}{dt} = F(t, \vec{x})$$

Where

$$\vec{x} = (x_1, x_2, ..., x_n)$$

$$\vec{F} = (F_1, F_2, ..., F_n)$$

An analytical study of systems is done in a Differential Equations course.

In this class, we will study them numerically

Cool Fact

A 2nd order Differential Equation.

Can be written as a system of 2 1st order Equations. This is useful, because we can use Euler's Method to solve numerically.

Ex.

$x" = -x$ Can be solved using techniques learned in Calc II and Differential Equations courses.

The general solution is $x(t) = A\cos t + B\sin t$

The particular solution satisfying $x(0) = 1$ and $x'(0) = 0$ is $x(t) = \cos t$

To solve this numerically we rewrite $x" = -x$ as a system.

To do this we just set $x_1 = x$ and $x_2' = x"$

So that the system is

$$x_1' = x_2$$
$$x_2' = -x_1$$

Using the code System-test.m

We can solve

$$x" = -x$$
$$x(0) = 1 \qquad x'(0) = 0$$

and we set Cosine as expected.

We also have the Lorenz.m Code for using Euler to solve a system of 3 equations.

We should be able to modify this code to solve the HH Model.

End of Lecture XIII

12 Lecture 14, July 9th

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} f(m_i)\Delta x$$

$$m_i = \frac{x_{i-1} + x_i}{2}$$

$$m_i \; \epsilon \; [x_{i-1}, \, x_i]$$

$$= \frac{b-a}{N}[f(m_1) + f(m_2) + ... + f(m_N)]$$

If a=-10, b=10:

$x_0 = -10$ $\qquad x = -10 : .1 : 10$ $\qquad Subinterval = .25$
$x_1 = -9.75$ $\qquad y = x.^2$ $\qquad xlength : x(201) - x(1)$

$steps = xlength * (\frac{1}{subint})$

Δx or $dx = \frac{xlength}{steps}$

$$m_1 = \frac{-10 + -9.75}{2}$$

$$mid = \frac{x(1) + x(2)}{2}$$

$for \;\; i = 1 : steps$

$mid = mid + .25$

midpoint-test.m

$a = -10;$

$b = 10;$

$N = 20;$ //Number of Rectangles

$x = a : .1 : b;$

$dx = \frac{b-a}{N}$; //Step Size.

$x = Zeros(N+1,1)$ //Storage for $x_0, x_1, x_2, ..., x_N$

$x(1) = a$; $x(2) = a + dx$

$m = Zeros(N,1)$; //Storage for midpoints

$m(1) = \frac{x(2)+x(1)}{2}$

$for\ i = 1 : N$
. $x(i+1) = x(i) + dx$;
. $m(i+1) = \frac{x(i)+x(i+1)}{2}$;
end

$F = m. * uparrow * 2$; //evals midpoints $y = x^2$

$Int = \Delta x * sum(f)$

$ff = x.^2$

$x_i = a + idx$ Left= $\Delta x[Sum(ff) - ff(x(N))]$ Right= $\Delta x[Sum(ff) - ff(x(1))]$

<u>FTC</u> $y'(x) = F(y, x)$

$$y(b) - y(a) = \int_a^b y'(t)dt$$

If, $a = x$, $b = x + h$

$$y(x+h) - y(x) = \int_x^{x+h} y'(t)dt$$

$$\approx y'(x^*) * h$$

$$y'(x) = F(y, x)$$

$$\int_a^b y'(x)dx = \int_a^b F(b, x)dx$$

$$y(b) - y(a) = \int_a^b F(y,x)dx$$

$$y(b) = y(a) + \int_a^b F(y,x)dx$$

Suppose limit $\lim_{n\to\infty} x_n = L$:

$$L = rL(1-L)$$

$$L = rL - rL^2$$

$$0 = rL - L - rL^2$$

$$0 = (r-1)L - rL^2$$

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(r-1) \pm \sqrt{(r-1)^2 - 4(-r)*0}}{2*(-r)}$$

$$= \frac{-(r-1) \pm r - 1}{-2r}$$

$$= \frac{-2(r-1)}{-2r}$$

$$= \frac{r-1}{r}$$

$$r = 2.5$$

$$L = \frac{2.5 - 1}{2.5} = .6$$

On computer you can put

$N = 10$
$r = .2$
$x(1) = .5$

$for\ i = 1 : N$
. $x(i+1) = x(i)$
end

ans=x'

$$x(2) = x(1) * (.2)(1 - x(1))$$

$$= .5(.2)(.5) = .05$$

$$x(3) = .05(.2)(1 - .05) = .0095$$

$$x(4) = .00188$$

$$x(5) = .000375$$

End of Lecture XIV

13 Lecture 15, July 11th

$$\frac{dy}{dt} = F(y,t)$$

$$y_{t+1} = y_t + \Delta t * F(y_t, t)$$

$$\frac{dy}{dt} = ty$$

$$\int \frac{1}{y} dy = \int t dt$$

$$lny = \frac{1}{2}t^2 + C$$

$$y = Ae^{\frac{1}{2}t^2}$$

Goal: Write own Euler scheme to solve.

$$\frac{dx}{dt} = \sigma(y - x)$$

In Euler below

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

$$x(i+1) = x(i) + dt[\sigma(y(i) - x(i))]$$

$$y(i+1) = y(i) + dt[x(i)(\rho - z(i)) - y(i)]$$

$$z(i+1) = z(i) + dt[x(i)y(i) - \beta z(i)]$$

$$t(i+1) = t(i) + dt$$

Solving Hodgkin-Huxley by Euler Scheme

$$y = \begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$y' = F(y,t)$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} . \\ UglyStuff \\ . \end{bmatrix}$$

$$\vec{y'} = F(\vec{y},t)$$

$$\vec{y}(i+1) = \vec{y}(i) + dt * F(\vec{y}(i),t(i))$$

$$\begin{bmatrix} y_1(i+1) \\ y_2(i+1) \\ y_3(i+1) \\ y_4(i+1) \end{bmatrix} = \begin{bmatrix} y_1(i) \\ y_2(i) \\ y_3(i) \\ y_4(i) \end{bmatrix} + dt \begin{bmatrix} F_1(y_1(i),t(i)) \\ F_2(y_2(i),t(i)) \\ . \\ . \end{bmatrix}$$

$$m = \begin{bmatrix} y_1(t_1) & y_2(t_1) & y_3(t_1) & y_4(t_1) \\ y_1(t_2) & y_2(t_2) & ... & ... \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$$

$$m = \begin{bmatrix} y(1,1) & y(2,1) & y(3,1) & y(4,1) \\ y(1,2) & y(2,2) & y(3,2) & y(4,2) \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$$

End of Lecture XV

14 Lecture 17, July 16th

Last class we solved the HH using a Euler Scheme. We feel good about it, because our picture is similar to pics from other code.

In general, the HH model is Difficult to Analyze.

Today we Introduce some ideas used to Analyze nonlinear equations and systems.

Recall: It is rare to be able to solve nonlinear equations and systems in closed form by hand.

This means it is rare to determine

A formula for the exact solution.

We do the next best thing:

Develop a theory so that we are confident our computed approximate solution is valid.

Strategy

Consider

$$x'(t) = F(t, x)$$

and study F.

If it has certain properties then the IVP has a unique solution. By fancy Fix Point Theorem.

Thus, we can study properties of the solution analytically and use them to validate numerically compute solutions.

To illustrate these ideas

Let's consider //another space to $x'(t)$

$$x'(t) = x(x-1) <==> \frac{dx}{dt} = x(x-1)$$

We can solve by separating variables.

But let's ignore that for the time being.

Instead, let's analyze the equation by 1st determining any Steady State Solutions

These are solutions x(t) satisfying

$$x'(t) = \frac{dx}{dt} = 0$$

For this equation

$$x'(t) = 0$$

$$<=> x(x-1) = 0$$

So that $x(t) = 0$ and $x(t) = 1$ are 2 Steady State Solutions.

Let's solve the equation precisely:

$$\frac{dx}{x(x-1)} = dt <=>$$

$$\int \frac{A}{x} + \frac{B}{x-1} dx = \int dt <=>$$

$$\int \frac{1}{x} + \frac{1}{x-1} dx = t + C$$

$$-ln(x) + ln(x-1) = t + C$$

$$ln(\frac{x-1}{x}) = t + C <=>$$

$$\frac{x-1}{x} = Ae^t <=>$$

$$x - 1 = Ae^t x <=>$$

$$x - Ae^t x = 1$$

$$x = \frac{1}{1 - Ae^t}$$

Let's compute sol. satisfying $x(0) = \frac{1}{2}$

$$x(0) = \frac{1}{1-A} <=> \frac{1}{2} = \frac{1}{1-A}$$

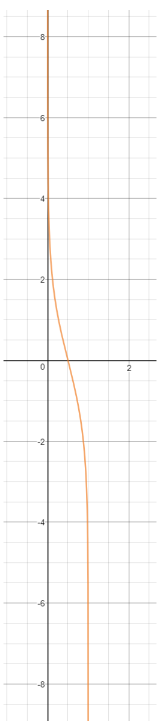
$$<=> 1 - A = 2$$

$$<=> -A = 1$$

Thus,

$$x(t) = \frac{1}{1 + e^t}$$

Let's Graph It!



<u>Note</u>: The solution is contained between the steady states

Let's plot some more

Say the solution passing through $(0, 2)$.

$$x(0) = 2 = \frac{1}{1 - Ae^0} <=>$$

$$2 = \frac{1}{1 - A} <=>$$

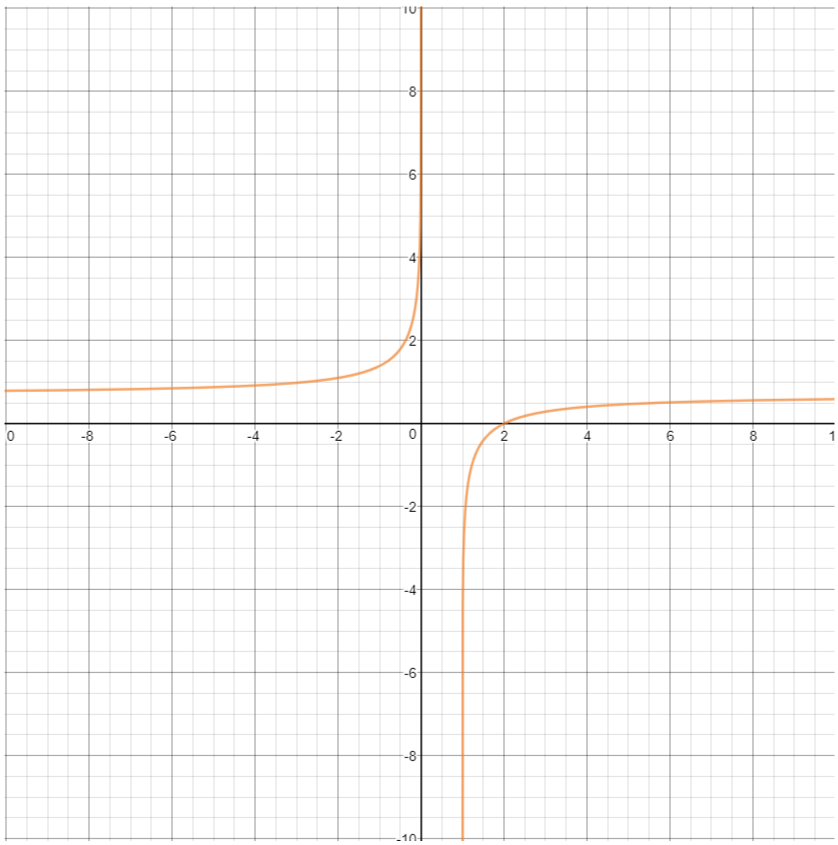
$$(1 - A)2 = 1 <=>$$

$$2 - 2A = 1 <=>$$

$$-2A = -1 <=>$$

$$A = \frac{1}{2}$$

$$x(t) = \frac{1}{1 - \frac{1}{2}e^t}$$



Note: The solution never crosses the steady states.

What about in a System?

Consider

$$x_1' = F_1(t, x_1, x_2)$$
$$x_2' = F_2(t, x_1, x_2)$$

and note the equilibrium sols satisfy
$0 = F_1$ and $0 = F_2$.

Suppose F_1 and F_2 do not depend on t explicitly:

$$F_1 = F_1(x_1, x_2) + F_2(x_1, x_2)$$

.
We can use Taylor's Theorem to linearize about the equilibrium solutions $\bar{x_1}$ and $\bar{x_2}$

For simplicity, assume $\bar{x_1} = \bar{x_2} = 0$

Then Taylor's Theorem says

$$F_1(x_1,x_2) \approx F_1(0,0) + F_{1x_1}(x_1,x_2) * x_1 + F_{1x_2}(x_1,x_2) * x_2$$

$$F(x,y) = x^2 + y^2 + xy$$

$$F_x(x,y) = 2x + 0 + y$$

$$F_y(x,y) = 0 + 2y + x$$

<u>Example</u>

$$\frac{dx}{dt} = 6x - 2x^2 - xy$$

$$\frac{dy}{dt} = 4y - xy - y^2$$

This is a <u>Non linear System</u>.

Let's compute equilibrium pts.

Clearly $(x,y) = (0,0)$ is one.

To find other solutions
set $x = 0$

$$0 = 0 * (6 - y)$$

$$0 = y(4 - y)$$

$$=> y = 0 \text{ or } y = 4$$

Thus, $(0,4)$ is also an equilibrium sol.

Similiarly, $y = 0$ gives $(3,0)$.

To use Taylors Theorem

Set

$$F_1 = 6x - 2x^2 - xy$$
$$F_2 = 4y - xy - y^2$$

and we differentiate:

$$F_{1x} = 6 - 4x - y$$

$$F_{1y} = -x$$

$$F_{2x} = -y$$

$$F_{2y} = 4 - x - 2y$$

Now evaluate at $(0, 0)$

$$F_{1x}(0,0) = 6 \qquad F_{2x}(0,0) = 0$$
$$F_{1y}(0,0) = 0 \qquad F_{2y}(0,0) = 4$$

Thus,

$$F_1(x,y) \approx F_1(0,0) + F_{1x}(0,0) * x + F_{1y}(0,0) * y$$

$$F_2(x,y) \approx F_2(0,0) + F_{2x}(0,0) * x + F_{2y}(0,0) * y$$

$$=> F_1(x,y) \approx 6x$$

$$F_2(x,y) \approx 4y$$

Our "linearized" system is

$$\frac{dx}{dt} = 6x$$

$$\frac{dy}{dt} = 4y$$

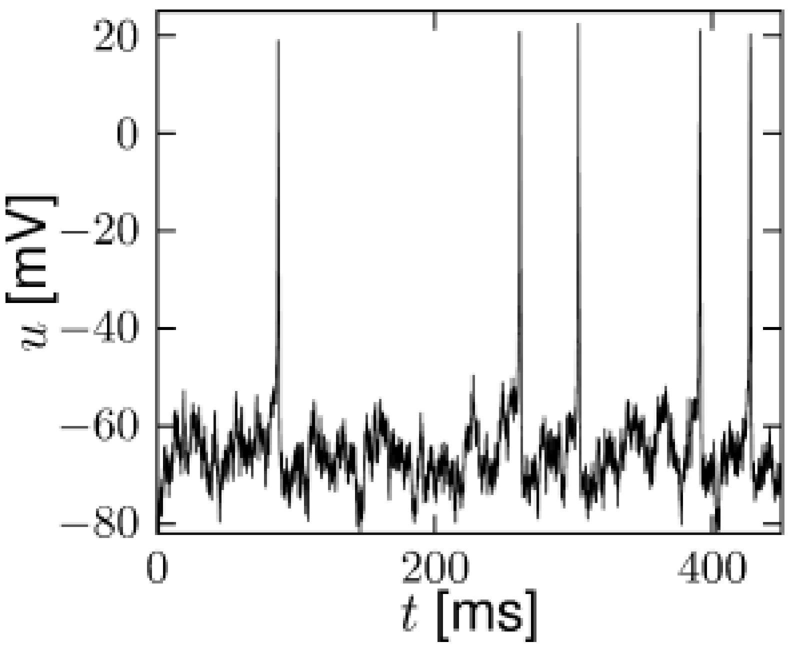
Other ways to analyze are:

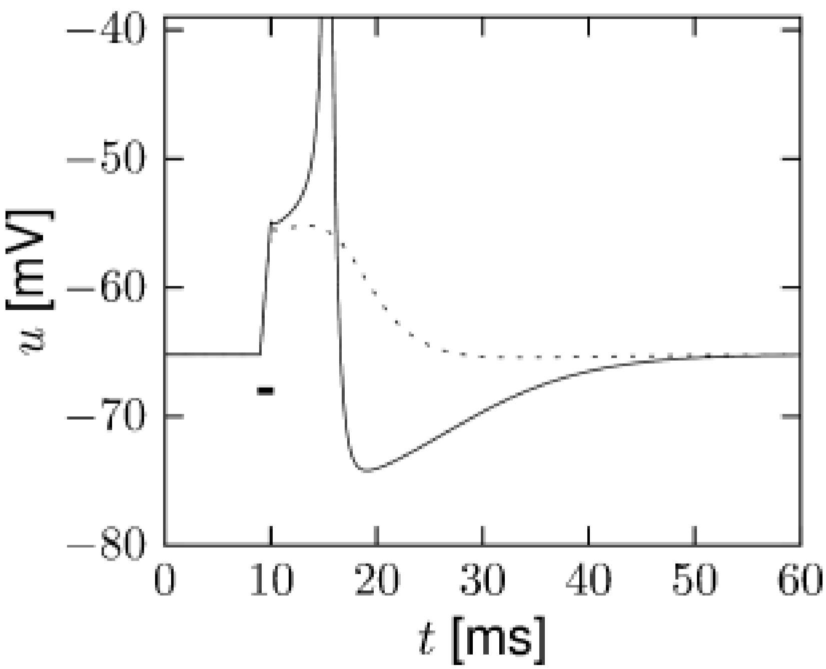
1. Phase Portraits
2. Slope Field //It can solve non linear equations
3. Study Asymptotics of equilibrium
4. and more

End of Lecture XVII

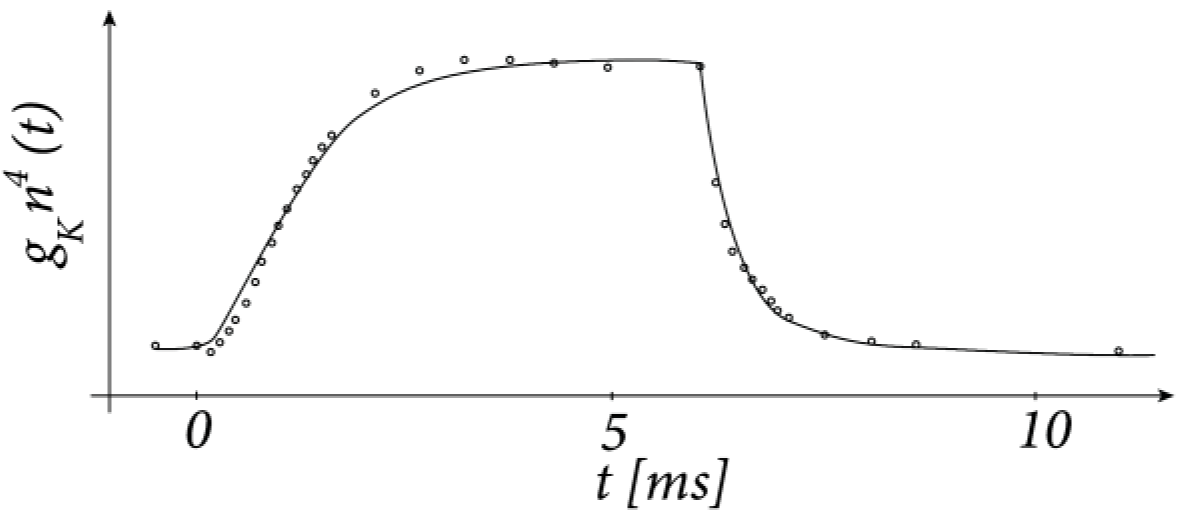
15 Lecture 19, July 18th

To motivate just imagine that in real life the voltage would more resemble





Than



The difference is that we expect the voltage to "hover" around its resting value as opposed to being the exact value all the time.

In mathematical terms we expect the voltage to be stochastic as opposed to deterministic.

One method to modeling this Feature is to use a Stochastic Differential Equation

Perhaps the most "basic" model is the Linear Integrate and Fire Model which we denote as IF.

$$\frac{dv}{dt} = \mu(t) + \sigma(t) * \frac{dw}{dt}$$

$\frac{dw}{dt}$ is the "random" or "Stochastic" term

The meaning of $\frac{dw}{dt}$ is well beyond the scope of this course.

If you're interested read up on a Wiener Process and Brownian Motion

A standard way of Analyzing a Stochastic Differential Equation is to use a Fokker Planck Equation which we will denote as FPE

This turns the question from

"What is the voltage at time t" to "What is the probability the voltage = blah at time t?"

In a special case we can Analyze the FPE by looking for a certain type of solution.

In doing this we arrive at a very useful algebraic equation:

$$\partial e^z = \partial \cos h(\partial) + z \sin h(\partial)$$

Where

$$z = \frac{\mu\theta}{\sigma^2} > 0$$

and

$$\partial = \partial_1 + i\partial_2$$

$$= \frac{\theta}{\sigma^2}\sqrt{\mu^2 + 2\lambda\sigma^2}$$

The Important numbers for solving the FPE are the values λ. They're called eigenvalues. In order to determine λ

We must solve $\star$.

It is possible that ∂ and hence λ are Complex Numbers.

Recall

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x^2 = \pm\sqrt{-1}$$
$$\pm i$$

Theorem $\star$ has infinitely many solutions and hence we can solve the FPE.

The proof is difficult, but today we will show how it can be done. The idea is common in Applied Mathematics

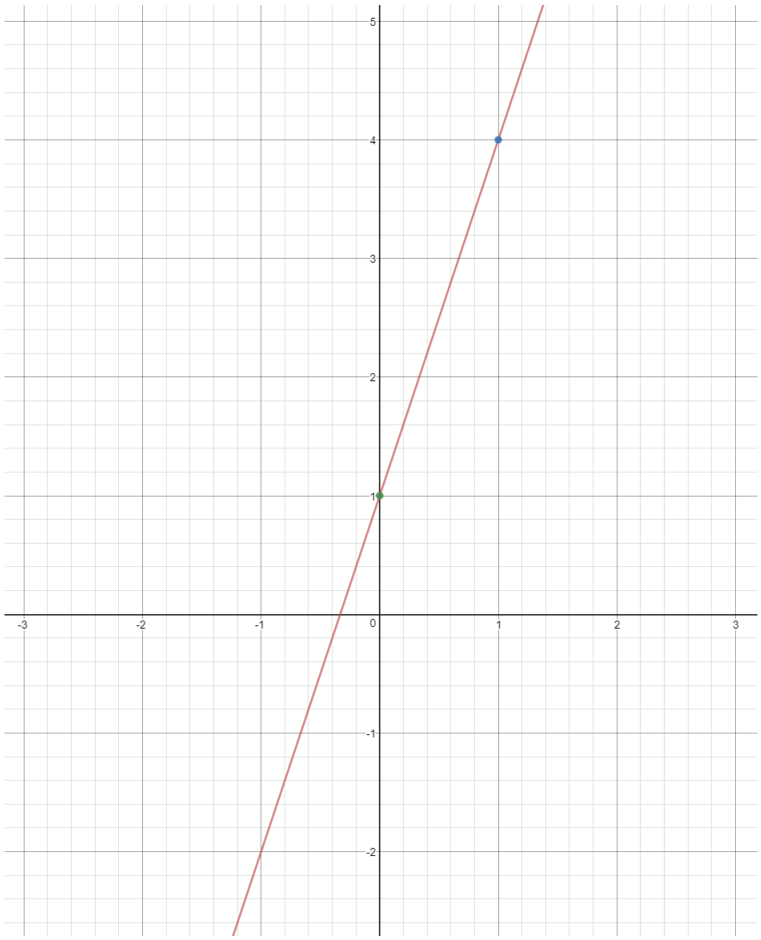
Solve an Algebraic Equation by turning it into Differential Equation(s) and Relying on Theory(Implicit Function Theorem).

Some equations are explicit and it's clear that there solutions.

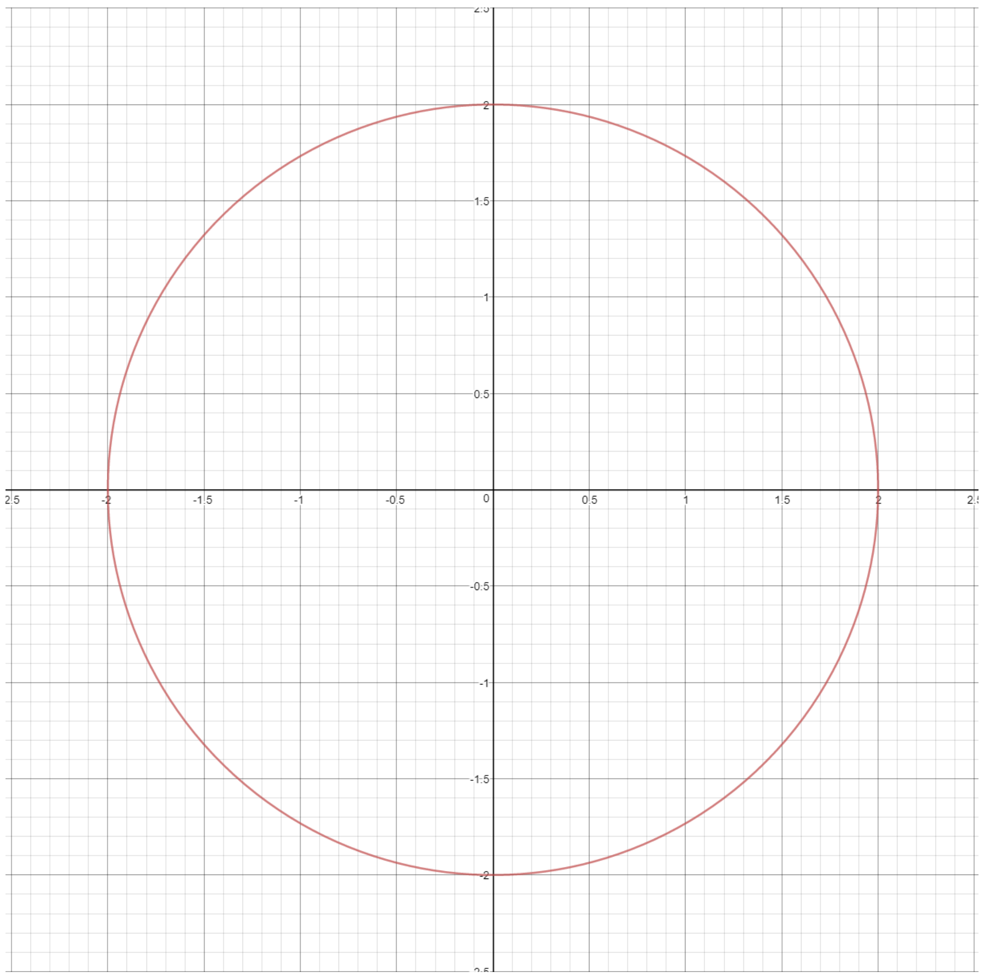
For example,

$$y = 3x + 1$$

Has infinitely many solutions. They can be represented by a line in the plane.



The equation $x^2 + y^2 = 4$ has Infinitely many solutions. They can be represented in the plane by a circle of Radius 2 centered at the origin.



However, if I just write

$$x^2y - 5\sin y + 4xy^3 = 0$$

It is unclear if there are any solutions.

A little guessing and checking shows (A,0) is a solution for any value of A.

Are there anymore??

So, if we plug the equation into Desmos it gives us a "crazy" graph. Is it correct?

One idea is to see if we can solve y in terms of x or x in terms of y.

Suppose we can solve $y = y(x)$

using <u>Implicit Diff(Chain Rule)</u> we might say

$$2xy + x^2 * y' - 5\cos y * y' + 4 * y^3 + 12y^2 * y' = 0$$

$$=> y' = \frac{-2xy - 4y^3}{x^2 - 5\cos y + 12y^2}$$

Note: $(0, -\pi)$ is also a solution.

This means we graph y(x) by solving the IVP

$$\begin{cases} y' = \frac{-2xy - 4y^3}{x^2 - 5\cos y + 12y^2} \\ y(0) = -\pi \end{cases}$$

This example is extremely challenging:

Who is correct: Desmos? Octave? Neither?

To illustrate with a simpler example consider

$x^2 + y^2 = 4$ which gives ODE

$$2x + 2y * y' = 0$$

$$<=> y' = \frac{-x}{y}$$

Moral: many complicated Algebraic equations can be "solved" numerically using a computer. However, we must have theoretical tools to validate our computations.

End of Lecture XIX

End of Intro to Mathematical Neuroscience